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ON FUNCTIONS ARISING AS POTENTIALS WITH OSCILLATING SYMBOLS

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Dedicated to Professor A.A. Kilbas on his 60th birthday

Abstract

We provide inversion formula for the potential type operator $\mathcal{M}^\alpha$ with symbol $|\xi|^{-\alpha}e^{i|\xi|}$ in the framework of the $L^p(\mathbb{R}^n)$ space, $1 \le p < \infty$, $\Re\alpha > 0$, and give conditions for $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ to be represented (in a sense of distributions, in general) as $\mathcal{M}^\alpha\varphi$ with some density $\varphi \in L^p(\mathbb{R}^n)$.

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1. Introduction

Nowadays there is a great deal of research devoted to the potential type operators - Riesz potential $\mathbb{I}^\alpha$ and its modifications (see [22], [23], [5], [6], [7], [8], [20], books [21], [10] and review papers [17], [18], [19]). The kernels of such operators have singularities at the origin and at infinity. Recall that the symbol of the Riesz potential $\mathbb{I}^\alpha$ is $\gamma_\alpha|\xi|^{-\alpha}$, where γ_α is some normalizing constant. Many aspects of such potential type operators theory are well studied.

In this paper we study the potential type operator $\mathcal{M}^\alpha$ with symbol $|\xi|^{-\alpha}e^{i|\xi|}$. It is obvious that when inserting oscillating function into multiplier, that is, when replacing the symbol $|\xi|^{-\alpha}$ with the symbol $|\xi|^{-\alpha}e^{i|\xi|}$, we essentially change the boundedness properties. The kernel $\Omega_\alpha(x) = (F^{-1}|\xi|^{-\alpha}e^{i|\xi|})(x)$ has singularities on the unit sphere in $\mathbb{T}^{n-1}$ in $\mathbb{R}^n$ and at infinity, and is smooth in $\mathbb{R}^n \setminus \mathbb{T}^{n-1}$. The investigation of such operators is a natural continuation of the above mentioned study of the Riesz potential and its modifications.

Note that operators with symbols $\chi(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}$, $0 \leqslant \alpha < n$ were considered in [15], [16] in connection to Cauchy problem for the wave equation. Here $\chi(r)$ is in $C^\infty(\mathbb{R}_+)$, and $\chi(r) = 0$ when $0 \leqslant r \leqslant 1$, $\chi(r) = 1$, when $r \geqslant 2$. In these papers it was described the set of pairs $(1/p, 1/q)$ for which the corresponding operator is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Similar result for the operator $\mathcal{M}^\alpha$ was established in [28] (see also [26]) for $\frac{n-1}{2} < \Re\alpha < n$, $1 \leq p < \frac{n}{\Re\alpha}$ and in [27] for $0 < \Re\alpha < n$, $1 \leq p < \frac{n}{\Re\alpha}$. More precisely, the ultimate paper treats general operators with symbols $a(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}$, where $a(|\xi|)$ is a fixed function satisfying some natural conditions. Below we cite boundedness results from [27] for $\mathcal{M}^\alpha$ for the sake of completeness.

The main results of this note are the following. We provide inversion formula for $\mathcal{M}^\alpha$ in the framework of $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $\Re\alpha > 0$. Further, we study the question when $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ is represented (in a sense of distributions, in general) as $\mathcal{M}^\alpha\varphi$ with some density $\varphi \in L^p(\mathbb{R}^n)$. Note that for $0 < \Re\alpha < n$, $1 \leq p < \frac{n}{\Re\alpha}$ the inversion of $\mathcal{M}^\alpha$ was studied in [24], though here we are not bounded by the mentioned range of parameters p, α. Note also that inversion problem for even more general operators was studied in [25]. The idea of this work is that in the considered case we can significantly simplify the corresponding constructions and use different methods to present new ideas in a more simpler and explicit form.

2. Preliminaries

Schwartz spaces of rapidly decreasing smooth functions in $\mathbb{R}^n$ will be denoted by $S(\mathbb{R}^n)$, and corresponding space of tempered distributions - by $S'(\mathbb{R}^n)$. We will also use special spaces of test functions and distributions introduced and studied by P.I. Lizorkin (see [3]). The P.I. Lizorkin space $\Phi(\mathbb{R}^n)$ of test functions consists of Schwartz functions φ, whose Fourier transform $\widehat{\varphi}$ vanishes at the origin along with all its derivatives. The dual space $\Psi(\mathbb{R}^n) = F\Phi(\mathbb{R}^n)$ is equipped with the countable set of norms

$$\|\psi\|_N = \sup_{\substack{x\in\mathbb{R}^n\setminus 0\\ |k|\leq N}} \left[\max\left\{\sqrt{1+|x|^2}, \frac{1}{|x|}\right\}\right]^N |\mathcal{D}^k\psi(x)|, \quad N = 0, 1, 2, \ldots. \tag{2.1}$$

where $\mathcal{D}^k\psi(x) = \frac{\partial^{|k|}}{\partial_{x_1}^{k_1}\ldots\partial_{x_n}^{k_n}}\psi(x)$, $k = (k_1, \ldots, k_n)$, $|k| = k_1 + \ldots + k_n$. Thus the space Φ can be equipped with the dual topology. The symbols $\Phi'(\mathbb{R}^n)$, $\Psi'(\mathbb{R}^n)$ stand for the spaces of distributions on $\Phi(\mathbb{R}^n)$, $\Psi(\mathbb{R}^n)$ correspondingly. Note, that (see [21], [11], [12]) the space $\Phi(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 < p < \infty$ and a Schwartz function $\varphi \in \Phi(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \varphi(x) x^k d\mu(x) = 0, \quad |k| = 0, 1, \ldots .$$

Here $x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$, $x = (x_1, x_2, \ldots, x_n)$, and $d\mu(x)$ stands for Lebesgue volume measure on $\mathbb{R}^n$.

THEOREM 2.1. ([21], [11]) *The function g, that corresponds to the distribution $g \in \Psi'(\mathbb{R}^n)$ is a multiplier in $\Psi(\mathbb{R}^n)$ if and only if $g \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and for each multi-index k there exist $\nu(k) \in \mathbb{N}$ and positive constant $c(k)$ such that*

$$|\mathcal{D}^k g(x)| \leqslant c(k) \left[\max \left\{ \sqrt{1 + |x|^2}, \frac{1}{|x|} \right\} \right]^{\nu(k)}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Let $\mathrm{W}_0(\mathbb{R}^n)$ denote Wienner ring, that is, the class of Fourier transforms of $L^1(\mathbb{R}^n)$ functions. Set $W_\varepsilon \varphi(x) = (W(\cdot, \varepsilon) * \varphi)(x)$, where $W(x, \varepsilon)$ stands for the Gauss-Weierstrass kernel: $W(x, \varepsilon) = (4\pi\varepsilon)^{-\frac{n}{2}} \exp\{-\frac{|x|^2}{4\varepsilon}\}$, $\varepsilon > 0$. Finally, set $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g(x)} d\mu(x)$.

3. The operator $\mathcal{M}^\alpha$ on functions $\varphi \in L^p(\mathbb{R}^n)$

The problem of inversion in the framework of $L^p(\mathbb{R}^n)$ and description of the image $\mathcal{M}^\alpha(L^p(\mathbb{R}^n))$ is tightly connected to establishing $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, and $L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ estimates for the corresponding potential operators. As it was noted, exact description of pairs $(1/p, 1/q)$ for which the operator $\mathcal{M}^\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $0 \le \Re\alpha < n$, $1 \le p \le q \le \infty$, was given in [27] (see also [26], [28] for the case $\frac{n-1}{2} < \Re\alpha < n$). For the sake of completeness, we provide here this result slightly reformulating it for our purposes.

THEOREM 3.1. ([27]) *Let $0 \leqslant \Re\alpha < n$. Then the operator $\mathcal{M}^\alpha$ is bounded:*

1. *from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p \leqslant q < \infty$ if and only if $\frac{1}{q} \leqslant \frac{1}{p} - \frac{\Re\alpha}{n}$ and either $\frac{1}{p} + \frac{1}{q} \leqslant 1$, $\frac{1}{p} - \frac{n}{q} \leqslant \Re\alpha - \frac{n-1}{2}$ or $\frac{1}{p} + \frac{1}{q} \geqslant 1$, $\frac{n}{p} - \frac{1}{q} \leqslant \Re\alpha + \frac{n-1}{2}$;*
2. *from $L^1(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 \leqslant q < \infty$ if and only if $\frac{n+1}{2} - \Re\alpha < \frac{1}{q} < 1 - \frac{\Re\alpha}{n}$;*
3. *from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ if and only if $\Re\alpha > \frac{n+1}{2}$ and $\Re\alpha = \frac{n+1}{2}$, $\Im\alpha \neq 0$;*
4. *from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, $1 < p < \infty$ if and only if $\frac{\Re\alpha}{n} < \frac{1}{p} < \Re\alpha - \frac{n-1}{2}$.*

The operator $\mathcal{M}^\alpha$ is of weak $(1, n/(n-\Re\alpha))$ - type for $n/2 \leqslant \Re\alpha < n$ and weak $(1, 1/((n+1)/2-\Re\alpha))$ - type for $n/2 \leqslant \Re\alpha < (n+1)/2$.

Although establishing $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ estimates is of an interests in its own right, it can be done only for a certain range of parameters (as in the theorem above). For a wider range of parameters we can provide $L^p(\mathbb{R}^n) \to L^r(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ estimates. This still would be sufficient for our further purposes – obtaining the inversion formula and describing the image $\mathcal{M}^\alpha(L^p(\mathbb{R}^n))$.

THEOREM 3.2. *Let $0 \leqslant \Re\alpha \leqslant \frac{n}{2}$, $1 \leqslant p < \frac{n}{\Re\alpha}$, $1 \leqslant r,\ s \leqslant \infty$. The operator $\mathcal{M}^\alpha$ is bounded:*

1. *from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $p \neq 1$ if $\frac{1}{s} \leqslant \frac{1}{p} - \frac{\Re\alpha}{n}$ and*
$$\frac{1}{p}+\frac{1}{r} \leqslant 1,\ \frac{1}{p}-\frac{n}{r} \leqslant \Re\alpha - \frac{n-1}{2};\ \ \frac{1}{p}+\frac{1}{r} \geqslant 1,\ \frac{n}{p}-\frac{1}{r} \leqslant \Re\alpha + \frac{n-1}{2}; \tag{3.1}$$
2. *from $L^1(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, if $\frac{n+1}{2} - \Re\alpha < \frac{1}{r} < 1$, $\frac{1}{s} < 1 - \frac{\Re\alpha}{n}$.*

P r o o f. Let as above $\chi(r)$ in $C^\infty(\mathbb{R}_+)$ and $\chi(r) = 0$ for $0 \leqslant r \leqslant 1$, $\chi(r) = 1$, for $r \geqslant 2$. Write $\mathcal{M}^\alpha$ in the form $\mathcal{M}^\alpha f = \mathcal{M}^\alpha_0 f + \mathcal{M}^\alpha_\infty f$, $f \in S(\mathbb{R}^n)$, where $\mathcal{M}^\alpha_0$, $\mathcal{M}^\alpha_\infty$ - operators with the symbols $m^0_\alpha(|\xi|) = \chi(|\xi|)|\xi|^{-\alpha}e^{i|\xi|}$, $m^\infty_\alpha(|\xi|) = (1-\chi(|\xi|))|\xi|^{-\alpha}e^{i|\xi|}$ respectively. The operator $\mathcal{M}^\alpha_0$ is bounded from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, $1 \leqslant p \leqslant r \leqslant \infty$, $r \neq 1$, $p \neq 1$, for $p,\ r$ satisfying (3.1). Indeed, Theorem 4.2 from [16] implies $\chi(|\xi|)|\xi|^{-\Re\alpha}e^{i|\xi|} \in M^r_p(\mathbb{R}^n)$ for such p, r. Since $|\xi|^{-i\Im\alpha} \in M^p_p(\mathbb{R}^n)$, $1 < p < \infty$, we have $m^0_\alpha(|\xi|) \in M^p_p(\mathbb{R}^n)$, $1 < p < \infty$ and $m^0_\alpha(|\xi|) \in M^r_p(\mathbb{R}^n)$, where
$$\frac{n+1}{2} - \Re\alpha < \frac{1}{r} < 1, \quad p = 1, \quad \frac{n-1}{2} < \Re\alpha \leqslant \frac{n}{2}, \tag{3.2}$$
and
$$\frac{n}{p} - \frac{1}{r} = \frac{n-1}{2} + \Re\alpha, \quad 1 < p \leqslant r \leqslant 2, \quad \Re\alpha \neq \frac{n}{2}. \tag{3.3}$$
Here we used the fact the translation invariant operator with the symbol $mn \in M^q_r(\mathbb{R}^n)$ is a composition of those with the symbols $m \in M^r_p(\mathbb{R}^n)$, $n \in M^q_r(\mathbb{R}^n)$ for either $p \leqslant r \leqslant q \leqslant 2$ or $2 \leqslant p \leqslant r \leqslant q$ (see [14]).

Due to convexity and duality arguments from (3.2), (3.3) we have that $\mathcal{M}^\alpha_0$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where p, q satisfy (3.1) with the exception $\frac{1}{q} > \frac{1}{p} - \frac{\Re\alpha}{n}$, but this case is covered by Theorem 3.1.

Consider $\mathcal{M}^\alpha_\infty$. Let first $\Re\alpha \neq \frac{n}{2}$ and $h(x)$ is a $C^\infty_0(\mathbb{R}^n)$ - function satisfying $h(x) = 1$, $|x| \leqslant 3$. According to the Sobolev theorem ([14]),

$|\xi|^{-\alpha} \in M_p^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{\Re\alpha}{n}$, and due to the Mikhlin multiplier theorem ([14], [4]) we have $h(\xi)e^{i|\xi|} \in M_p^p(\mathbb{R}^n)$, $1 < p < \infty$. As above, we have (at first for $q \leqslant 2$ and then by duality for all q) that $m_\alpha^\infty(|\xi|) \in M_r^q(\mathbb{R}^n)$, $r \leqslant p$, and $m_\alpha^\infty(|\xi|) \in M_p^s(\mathbb{R}^n)$, $s \geqslant q$, where $\frac{1}{q} = \frac{1}{p} - \frac{\Re\alpha}{n}$.

Let now $\Re\alpha = \frac{n}{2}$. Since $m_\alpha^\infty(|\xi|) \in L^p(\mathbb{R}^n)$, $1 \leqslant p < 2$, applying the Young-Hausdorff inequality: $\|Fu\|_q \leqslant (2\pi)^{\frac{n}{2q}} \|u\|_p$, $1 \leqslant p \leqslant 2$, $\frac{1}{p} + \frac{1}{q} = 1$, we obtain that $F^{-1}m_\alpha^\infty \in L^s(\mathbb{R}^n)$, $s > 2$, and hence the operator $\mathcal{M}_\infty^\alpha$ is bounded from $L^1(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$, $s > 2$. The theorem is now proved. ∎

Given $0 \leqslant \Re\alpha < n$ let the symbol $\Delta(\alpha)$ stand for the set of p, $1 \leqslant p \leqslant \infty$, such that the operator $\mathcal{M}^\alpha$ is bounded either form $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ or from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ according to Theorems 3.1, 3.2. So,

$$\Delta(\alpha) = \begin{cases} \{p \in [1,\infty] : \frac{\Re\alpha}{n} < \frac{1}{p} \leqslant 1\}, & \frac{n-1}{2} < \Re\alpha < n, \\ \{p \in [1,\infty] : \frac{\Re\alpha}{n} < \frac{1}{p} < \frac{1}{2} + \frac{\Re\alpha}{n-1}\}, & \frac{n(n-1)}{2(2n-1)} \leqslant \Re\alpha \leqslant \frac{n-1}{2}, \\ \{p \in [1,\infty] : \frac{1}{2} - \frac{\Re\alpha}{n-1} \leqslant \frac{1}{p} \leqslant \frac{1}{2} + \frac{\Re\alpha}{n-1}\}, & 0 \leqslant \Re\alpha < \frac{n(n-1)}{2(2n-1)}. \end{cases}$$

Now for $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$ the operator $\mathcal{M}^\alpha$ (defined on $\mathrm{S}(\mathbb{R}^n)$) can be extended for $\varphi \in L^p(\mathbb{R}^n)$.

For either $0 \leqslant \Re\alpha < n$, $p \notin \Delta(\alpha)$, or $\Re\alpha \geqslant n$, $1 \leqslant p \leqslant \infty$ we can also define the operator $\mathcal{M}^\alpha$ on $L^p(\mathbb{R}^n)$ treating it in the sense of $\Phi'(\mathbb{R}^n)$ distribution and considering $\varphi \in L^p(\mathbb{R}^n)$ as element of $\Phi'(\mathbb{R}^n)$. Set $\langle \mathcal{M}^\alpha\varphi, \omega\rangle = \langle \varphi, (\mathcal{M}^\alpha)^*\omega\rangle$, $\omega \in \Phi(\mathbb{R}^n)$. Here $(\mathcal{M}^\alpha)^*\omega = \overline{\mathcal{M}^\alpha\overline{\omega}} \equiv \mathcal{M}^{\overline{\alpha}}\omega$. This definition is correct: since $|\xi|^\alpha e^{i|\xi|}$ is a multiplier in $\Psi(\mathbb{R}^n)$, then according to the Gelfand-Shilov theorem ([1]) the distribution $\Omega_\alpha \equiv F^{-1}|\xi|^\alpha e^{i|\xi|}$ is a convolutor in $\Phi(\mathbb{R}^n)$ (that is the mapping $\omega \in \Phi(\mathbb{R}^n) \to \Omega_\alpha * \omega \in \Phi(\mathbb{R}^n)$ is continuous in $\Phi(\mathbb{R}^n)$).

Note that for $(n+1)/2 < \Re\alpha < n$ the operator $\mathcal{M}^\alpha$ is the integral operator

$$\mathcal{M}^\alpha f(x) = \int_{\mathbb{R}^n} \Omega_\alpha(y)\ f(x-y)\ d\mu(y), \quad f \in \mathrm{S}(\mathbb{R}^n),$$

with the kernel

$$\Omega_\alpha(x) = \frac{i^{n-\alpha}\Gamma(n-\alpha)}{2^{n-1}\pi^{n/2}\Gamma(n/2)}\ \mathrm{F}\left(\frac{n-\alpha}{2}, \frac{n-\alpha+1}{2}; \frac{n}{2}; |x|^2\right), \tag{3.4}$$

where $\mathrm{F}(a,b;c;d)$ is the Gauss hypergeometric function. For $|x| > 1$ the right-hand side in (3.4) is understood as analytic continuation. To calculate the kernel, we apply the Bochner formula for the Fourier transform of a radial function (see [10], p. 485). For the remained "regular" values of α $(0 \leq \Re\alpha < n)$ we refer to the analytic continuation of the above integral representation, given in [27].

4. Inversion of $\mathcal{M}^\alpha$ in $L^p(\mathbb{R}^n)$

Let

$$k_{\alpha,\varepsilon}(x) = F^{-1}\bigg(|\xi|^\alpha e^{-i|\xi|}\exp\{-\varepsilon|\xi|^2\}\bigg)(|x|). \tag{4.1}$$

In the framework of approximative operators method (see [2]), set

$$\mathcal{K}^\alpha f = \lim_{\varepsilon\to 0}\mathcal{K}^\alpha_\varepsilon f \equiv \lim_{\varepsilon\to 0} k_{\alpha,\varepsilon} * f. \tag{4.2}$$

The limit in (4.2) is understood either in the $L^p(\mathbb{R}^n)$ - norm, or in the sense of almost everywhere convergence.

Note that the function under the Fourier transform in the right-hand side (4.1) (redefined at the origin by continuity) belongs to the Wienner ring $\mathrm{W}_0(\mathbb{R}^n)$ (use results from [13]). This function also belongs to $L^1(\mathbb{R}^n)$. Hence, $k_{\alpha,\varepsilon} \in L^1(\mathbb{R}^n)$.

LEMMA 4.1. *Let $\Re\alpha \geqslant 0$, $1 \leqslant p \leqslant \infty$, $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $1 \leqslant r, s \leqslant \infty$. If $f = \mathcal{M}^\alpha\varphi$, $\varphi \in L^p(\mathbb{R}^n)$ almost everywhere or in the sense of the $\Phi'(\mathbb{R}^n)$ - distributions, then*

$$\mathcal{K}^\alpha_\varepsilon f = W_\varepsilon\varphi. \tag{4.3}$$

P r o o f. Applying the Fubini theorem we have $\langle\mathcal{K}^\alpha_\varepsilon f,\omega\rangle = \langle f,\overline{\mathcal{K}^\alpha_\varepsilon\overline{\omega}}\rangle = \langle f,\overline{k}_{\alpha,\varepsilon} * \omega\rangle$, $\omega \in \Phi(\mathbb{R}^n)$. Since $k_{\alpha,\varepsilon} \in L^1(\mathbb{R}^n)$, there exists a sequence $k_\delta \in \mathrm{S}(\mathbb{R}^n)$ such that $\|\overline{k}_{\alpha,\varepsilon} - k_\delta\|_1 \to 0$, when $\delta \to 0$. Obviously, $k_\delta * \omega \in \Phi(\mathbb{R}^n)$ for any $\omega \in \Phi(\mathbb{R}^n)$. Hence,

$$\begin{aligned}\langle f,\overline{k}_{\alpha,\varepsilon} * \omega\rangle &= \lim_{\delta\to 0}\langle f, k_\delta * \omega\rangle = \lim_{\delta\to 0}\langle\mathcal{M}^\alpha\varphi, k_\delta * \omega\rangle = \lim_{\delta\to 0}\langle\varphi,\overline{\mathcal{M}^\alpha(\overline{k_\delta * \omega})}\rangle\\ &= \lim_{\delta\to 0}\langle\varphi, k_\delta * \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle = \langle\varphi,\overline{\mathcal{K}^\alpha_\varepsilon\mathcal{M}^\alpha\overline{\omega}}\rangle = \langle\varphi, W_\varepsilon\omega\rangle = \langle W_\varepsilon\varphi,\omega\rangle,\ \omega \in \Phi(\mathbb{R}^n).\end{aligned}$$

Therefore,

$$\mathcal{K}^\alpha_\varepsilon f \overset{(\Phi'(\mathbb{R}^n))}{=} W_\varepsilon\varphi.$$

Since two locally summable functions (distributions form $\mathrm{S}'(\mathbb{R}^n)$), that coincide in $\Phi'(\mathbb{R}^n)$, may differ only by a polynomial, we have $\mathcal{K}^\alpha_\varepsilon f(x) = W_\varepsilon\varphi(x)$ almost everywhere $x \in \mathbb{R}^n$. Here we noted that $\mathcal{K}^\alpha_\varepsilon f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $W_\varepsilon\varphi \in L^p(\mathbb{R}^n)$, and hence cannot "contain" a polynomial. The proof is complete. ■

REMARK 4.2. If $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$, then (4.3) is true for $\varphi \in L^p(\mathbb{R}^n)$, that is, $\mathcal{K}^\alpha_\varepsilon\mathcal{M}^\alpha\varphi = W_\varepsilon\varphi$, $\varphi \in L^p(\mathbb{R}^n)$.

According to above said we establish the following inversion formula.

THEOREM 4.3. *Let* $\Re\alpha \geqslant 0$, $1 \leqslant p \leqslant \infty$, $\varphi \in L^p(\mathbb{R}^n)$, $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $1 \leqslant r, s \leqslant \infty$. *If* $f(x) = \mathcal{M}^\alpha\varphi(x)$ *almost everywhere for* $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$ *or* $f = \mathcal{M}^\alpha\varphi$ *in the sense of* $\Phi'(\mathbb{R}^n)$ *distributions, then*

$$\mathcal{K}^\alpha f = \varphi, \quad \varphi \in L^p(\mathbb{R}^n),$$

where the operator $\mathcal{K}^\alpha$ *is given by (4.2) and the limit in (4.2) is understood in the* $L^p(\mathbb{R}^n)$*-norm for* $p \neq \infty$, *or in the sense of almost everywhere convergence.*

P r o o f. It is known that $W_\varepsilon\varphi \to \varphi$, $\varphi \in L^p(\mathbb{R}^n)$, $\varepsilon \to 0$ in $L^p(\mathbb{R}^n)$ norm for $1 \leqslant p < \infty$ and in the sense of almost everywhere convergence for $1 \leqslant p \leqslant \infty$. To end the proof, it is sufficient to refer to Lemma 4.1. ■

5. Representation of f in $L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ as $\mathcal{M}^\alpha\varphi$

In the next theorem we give sufficient conditions for a function $f \in L^r(\mathbb{R}^n)+L^s(\mathbb{R}^n)$ to be represented (in a sense of distributions, in general) as $\mathcal{M}^\alpha\varphi$ with some $\varphi \in L^p(\mathbb{R}^n)$. In some sense, this result is inverse to Theorem 4.3 and immediately provides the characterization of $\mathcal{M}^\alpha(L^p(\mathbb{R}^n))$, if the operator $\mathcal{M}^\alpha$ is understood in the regular sense, and the characterization of $\mathcal{M}^\alpha(L^p(\mathbb{R}^n)) \cap L^q(\mathbb{R}^n)$ (more general $\mathcal{M}^\alpha(L^p(\mathbb{R}^n)) \cap (L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)))$ otherwise.

THEOREM 5.1. *Let* $\Re\alpha \geqslant 0$, $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $1 \leqslant r, s \leqslant \infty$.

1. *If* $\mathcal{K}^\alpha_\varepsilon f$ *is convergent in* $L^p(\mathbb{R}^n)$ *norm when* $\varepsilon \to 0$, $1 \leqslant p < \infty$, *then* $f \overset{(\Phi'(\mathbb{R}^n))}{=} \mathcal{M}^\alpha\varphi$, *where* $\varphi = \lim^{(L^p(\mathbb{R}^n))}_{\varepsilon\to 0} \mathcal{K}^\alpha_\varepsilon f$.
2. *If* $\sup_{\varepsilon>o} \|\mathcal{K}^\alpha_\varepsilon f\|_p < \infty$, $1 < p < \infty$, *then there exists* $\varphi \in L^p(\mathbb{R}^n)$ *such that* $f \overset{(\Phi'(\mathbb{R}^n))}{=} \mathcal{M}^\alpha\varphi$.
3. *For* $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$ *the equality* $f = \mathcal{M}^\alpha\varphi$ *in 2 and 3 above takes place almost everywhere in* $\mathbb{R}^n$.

P r o o f. Let $f \in L^r(\mathbb{R}^n)+L^s(\mathbb{R}^n)$ and $\mathcal{K}^\alpha f = \varphi \in L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$. Then for any $\omega \in \Phi(\mathbb{R}^n)$ we have:

$$\begin{aligned}\langle \mathcal{M}^\alpha\varphi, \omega\rangle &= \langle \varphi, \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle = \langle \overset{(L^p(\mathbb{R}^n))}{\lim_{\varepsilon\to 0}} \mathcal{K}^\alpha_\varepsilon f, \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle = \lim_{\varepsilon\to 0}\langle \mathcal{K}^\alpha_\varepsilon f, \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle \\ &= \lim_{\varepsilon\to 0}\langle f, \overline{\mathcal{K_\varepsilon}^\alpha\mathcal{M}^\alpha\overline{\omega}}\rangle = \langle f, W_\varepsilon\omega\rangle.\end{aligned}$$

The third equality follows from the fact that convergence in $L^p(\mathbb{R}^n)$ implies that in $\Phi'(\mathbb{R}^n)$. Now for $\omega \in \Phi(\mathbb{R}^n)$, we obtain

$$W_\varepsilon\omega \overset{(\Phi(\mathbb{R}^n))}{\longrightarrow} \omega, \quad \varepsilon \longrightarrow 0.$$

The convergence of the Fourier images in the dual space $\Psi(\mathbb{R}^n)$ implies

$$\exp\{-\varepsilon|\xi|^2\}\widehat{\omega}(\xi) \overset{(\Psi(\mathbb{R}^n))}{\longrightarrow} \widehat{\omega}(\xi), \quad \varepsilon \longrightarrow 0,$$

which can be checked directly using the definition of topology in $\Psi(\mathbb{R}^n)$. Hence for $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ we have $\langle f, W_\varepsilon\omega\rangle \longrightarrow \langle f, \omega\rangle$ when $\varepsilon \to 0$. Therefore,

$$\langle \mathcal{M}^\alpha\varphi, \omega\rangle = \langle f, \omega\rangle, \quad \omega \in \Phi(\mathbb{R}^n). \tag{5.1}$$

Let $f \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n)$, $\sup_{\varepsilon > o} \|\mathcal{K}^\alpha_\varepsilon f\|_p < \infty$. We have

$$|\langle \mathcal{K}^\alpha_\varepsilon f, \omega\rangle| \leqslant c\|\omega\|_{p'}, \quad 1/p + 1/q,$$

and due to the weak compactness of $L^p(\mathbb{R}^n)$, $1 < p < \infty$ there exist a sequence ε_k and a function $\varphi \in L^p(\mathbb{R}^n)$ such that

$$\langle \mathcal{K}^\alpha_{\varepsilon_k} f, \omega\rangle \longrightarrow \langle \varphi, \omega\rangle, \quad \varepsilon \longrightarrow 0, \quad \omega \in L^{p'}(\mathbb{R}^n),$$

where as usual $\frac{1}{p'} = 1 - \frac{1}{p}$. For mentioned φ and for $\omega \in \Phi(\mathbb{R}^n)$ we have

$$\langle \mathcal{M}^\alpha\varphi, \omega\rangle = \langle \varphi, \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle = \lim_{\varepsilon_k \to 0} \langle \mathcal{K}^\alpha_{\varepsilon_k} f, \overline{\mathcal{M}^\alpha\overline{\omega}}\rangle = \lim_{\varepsilon_k \to 0} \langle f, \mathcal{K}^\alpha_{\varepsilon_k}\mathcal{M}^\alpha\omega\rangle = \langle f, \omega\rangle.$$

Hence (5.1) is true in this case as well.

For $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$, $\varphi \in L^p(\mathbb{R}^n)$ according to Theorems 3.1, 3.2, we obtain $\mathcal{M}^\alpha\varphi \in L^q(\mathbb{R}^n) + L^s(\mathbb{R}^n)$ with some q, $s \in [p, \infty]$. Now from $\mathcal{M}^\alpha\varphi \overset{(\Phi'(\mathbb{R}^n))}{=} f$ we have $\mathcal{M}^\alpha\varphi(x) = f(x)$ for almost all $x \in \mathbb{R}^n$ (analogously to the proof of Lemma 4.1). The proof is complete. ■

Let $\mathcal{M}^\alpha(L^p(\mathbb{R}^n))$ $(\subset \Phi'(\mathbb{R}^n))$ stand for the image of $L^p(\mathbb{R}^n)$ under the action of $\mathcal{M}^\alpha$.

Theorem 5.2. *Let $0 \leqslant \Re\alpha < n$, $p \in \Delta(\alpha)$, $p \neq 1$, $1 \leqslant r, s \leqslant \infty$. Then*

$$\begin{aligned}\mathcal{M}^\alpha(L^p(\mathbb{R}^n)) &= \{f(x) \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n) : \mathcal{K}^\alpha f(x) \in L^p(\mathbb{R}^n)\}\\ &= \{f(x) \in L^r(\mathbb{R}^n) + L^s(\mathbb{R}^n) : \sup_{\varepsilon>0} \|\mathcal{K}^\alpha_\varepsilon f(x)\|_p < \infty\},\end{aligned}$$

where $\mathcal{K}^\alpha f = \lim^{(L^p(\mathbb{R}^n))}_{\varepsilon \to 0} \mathcal{K}^\alpha_\varepsilon f$.

P r o o f. Follows by applying Theorems 4.3, 5.1 according to (4.3). ■

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